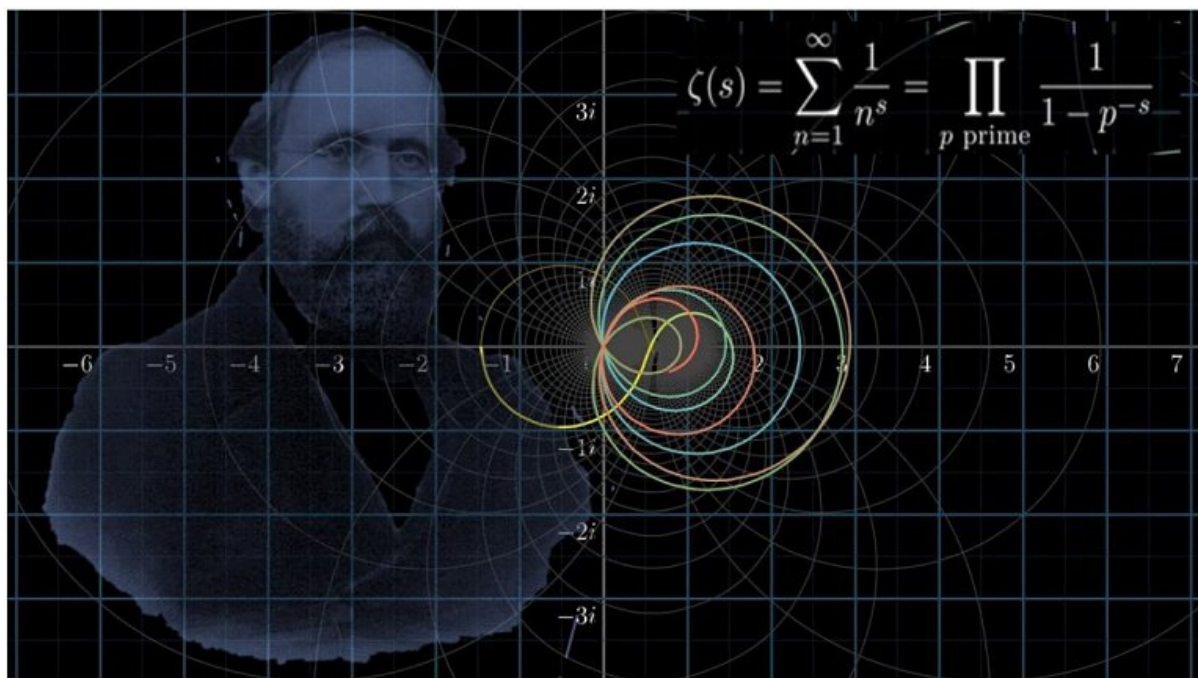


# Evidence of equivalent conditions for the Riemann Hypothesis



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## Abstrakt / Abstract:

Eng: Searching for evidence of equivalent conditions for RH (Riemann's hypothesis).  
Equivalent conditions have been created in the past by the authors: Srinivasa Ramanujan (Ramanujan), Lagarias, Gronwall, Robin (see references (1), (2)).

# Preface

This publication is devoted to finding evidence in favor of the Riemann hypothesis. It follows up directly on the articles with the reference (7), (8), (9). Articles with the reference (7), (8), (9) contain a sequence of my thoughts on how to find evidence of equivalent conditions for RH, as Srinivasa Ramanujan (Ramanujan), Lagarias, Gronwall, Robin did in the past (see references (1), (2).)). This publication should be read in the context of the articles - references (7), (8), (9). In the publication, I will gradually present the steps and procedures that lead to the proof of equivalent conditions for RH.

NOTE:  $\log x = \ln x$ ,  $\log x$  - the natural logarithm.

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Reference (1),(2),(5).

citation:

The sum-of-divisors function  $\sigma$  is defined by

$$\sigma(n) := \sum_{d|n} d$$

For example,  $\sigma(4) = 7$  and  $\sigma(pn) = (p + 1)\sigma(n)$ , if  $p$  is a prime not dividing  $n$ . In 1913, the Swedish mathematician Thomas Gronwall found the maximal order of  $\sigma$ .

Theorem 1. (Gronwall) The function

$$G(n) := \frac{\sigma(n)}{n \log(\log n)}$$

satisfies  $\lim_{n \rightarrow \infty} \sup G(n) = e^{\gamma} = 1.78107\dots$ , where  $\gamma$  is the Euler-Mascheroni constant.

Theorem 2. (Ramanujan) If the Riemann Hypothesis is true, then

$$G(n) < e^{\gamma} \quad (n \gg 1)$$

Here  $n \gg 1$  means for all sufficiently large  $n$ . In 1984, the French mathematician Guy Robin proved that a stronger statement about the function  $G$  is equivalent to the RH.

Theorem 3. (Robin) The Riemann Hypothesis is true if and only if

$$G(n) < e^{\gamma}; \quad (n > 5040).$$

Theorem 4. (Lagarias) The Riemann Hypothesis is true if and only if

$$\sigma(n) < Hn + \exp(Hn) \log(Hn) \quad (n > 1)$$

where  $Hn$  denotes the  $n$ th harmonic number  $Hn$  :

$$Hn = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$$

$$\sum_{n=1}^k \frac{1}{n} = \ln k + \gamma + \varepsilon_k \leq \ln k + 1; \quad \varepsilon_k \sim \frac{1}{2k}$$

## Numerical tests

Numerical tests are described in detail in the article - reference (7), and advanced tests - reference (8).

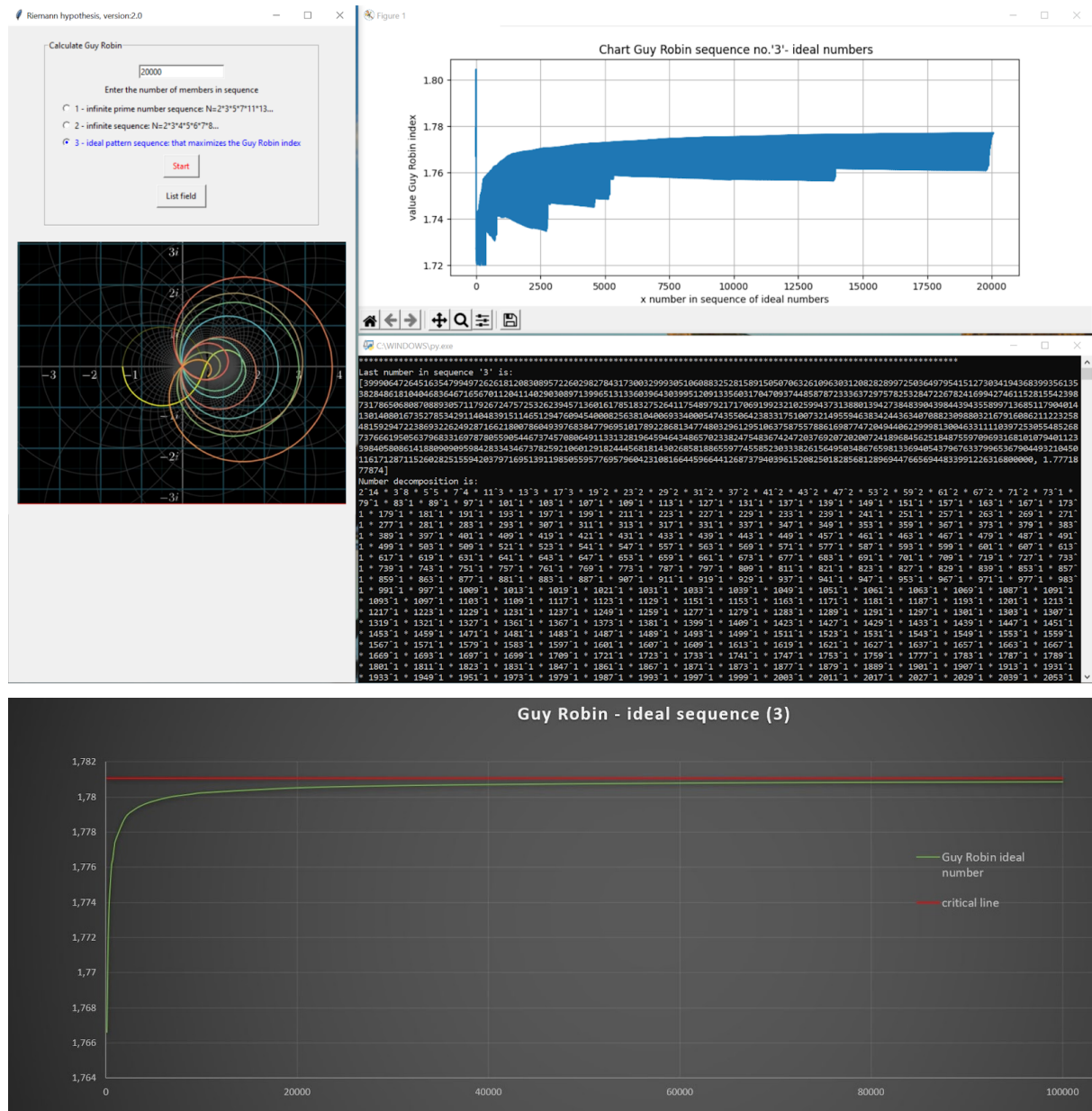


Fig.1 Guy Robin test for smaller and advanced testing.

Note: ideal numbers are called highly composite numbers in the literature.

Final verzia Riemann / Final version of Riemann.

referencia (10) / reference (10):

download file in Pythone - Github : [riemannhypothesis\\_final.py](https://github.com/riemannhypothesis/riemannhypothesis_final.py)

DESCRIPTION: The algorithm contains a calculation of the Guy Robin equation for different sequences, to verify the validity of the Riemann hypothesis. Optimally, the calculation for a sequence of 3: 40000 ideal numbers takes about 20 minutes. Download the image for riemann.png from github (if you are running the code in python), place it in the same folder as riemann\_hypothesis\_final.py, otherwise it will display an error message.

### Advanced testing

Riemann test version (program source code):

download file in Pythone - Github: [Riemanm\\_test.py](#)

### Results:

The value of  $e^\gamma$  was not exceeded. The test was performed only for highly-composite numbers consisting of the first 500 thousand prime numbers multiplied by each other. The number "N" had a value of up to 3201675 digits.

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## Sigma calculation

It appears in the relations. for  $G(n)$ :

$G(n)$ :

example:

$$n = 12; \sigma(n) = 1 + 2 + 3 + 4 + 6 + 12 = 28$$

each number can be decomposed, prime decomposition:

$$n = \prod_{i,j} p_i^{j_i}; n = p_1^{j_1} \cdot p_2^{j_2} \cdot p_3^{j_3} \dots p_n^{j_n}; p_i \in \text{prime}; j_i \in N \quad (1.1)$$

example:

$$n = 2 \cdot 3 \cdot 5 \cdot 7$$

$$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$$

let's define a simple sequence (1):

$$n = \prod_i^n p_i; n = p_1 \cdot p_2 \cdot p_3 \dots p_n; p_i \in \text{prime} \quad (1.2)$$

then it applies:

$$\sigma(n) = \prod_{p_i \in \text{prime}}^{p_n} (p_i + 1) \quad (1.3)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in \text{prime}}^{p_n} \left( \frac{p_i + 1}{p_i} \right) = \prod_{p_i \in \text{prime}}^{p_n} \left( 1 + \frac{1}{p_i} \right) \quad (1.4)$$

example:

$$n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$$

$$\sigma(n) = (2 + 1) \cdot (3 + 1) \cdot (5 + 1) \cdot (7 + 1) \cdot (11 + 1) \cdot (13 + 1) = 96768$$

for (1.1) is  $\sigma(n)$ :

$$\sigma(n) = \prod_{p_i \in \text{prime}}^{p_n} (1 + p_i + p_i^2 + p_i^3 + \dots + p_i^{j_i}) \quad (1.5)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in \text{prime}}^{p_n} \frac{(1 + p_i + p_i^2 + p_i^3 + \dots + p_i^{j_i})}{p_i^{j_i}} \quad (1.6)$$

example:

$$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$$

$$\sigma(n) = (1 + 2^1 + 2^2 + 2^3 + 2^4) \cdot (1 + 3^1 + 3^2) \cdot (5 + 1) \cdot (7 + 1) = 19344$$

## Highly composite numbers

Let's define a sequence (3) - highly composite numbers:

highly composite numbers are numbers that maximize  $\sigma(n)/n$ .

$$\sup \frac{\sigma(n)}{n} = \sup \prod_{p_i \in \text{prime}}^{p_n} \frac{(1+p_i+p_i^2+p_i^3+\dots+p_i^{j_i})}{p_i^{j_i}} \quad (1.7)$$

example:

$$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 720720$$

$$\sigma(n) = 3249792; \sigma(n)/n = 4.509..$$

let's adjust the equation (1.5):

reference (2), page 9

$$\sigma(n) = \prod_{p_i \in \text{prime}}^{p_n} (1 + p_i + p_i^2 + p_i^3 + \dots + p_i^{j_i}) = \prod_{p_i \in \text{prime}}^{p_n} \frac{p_i^{j_i+1}-1}{p_i-1} \quad (1.8)$$

we substitute into the equation (1.7):

$$\sup \frac{\sigma(n)}{n} = \sup \prod_{p_i \in \text{prime}}^{p_n} \frac{p_i^{j_i+1}-1}{(p_i-1) p_i^{j_i}} \quad (1.9)$$

$$\frac{\sigma(n)}{n} = (2 - 2^{-j_2})(\frac{3}{2} - \frac{3^{-j_3}}{2})(\frac{5}{4} - \frac{5^{-j_5}}{4})(\frac{7}{6} - \frac{7^{-j_7}}{6})\dots \quad (1.10)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in \text{prime}}^{p_n} (\frac{p_i}{p_i-1} - \frac{p_i^{-j_i}}{p_i-1}) \quad (2.0)$$

let's define  $\beta(n)$ :

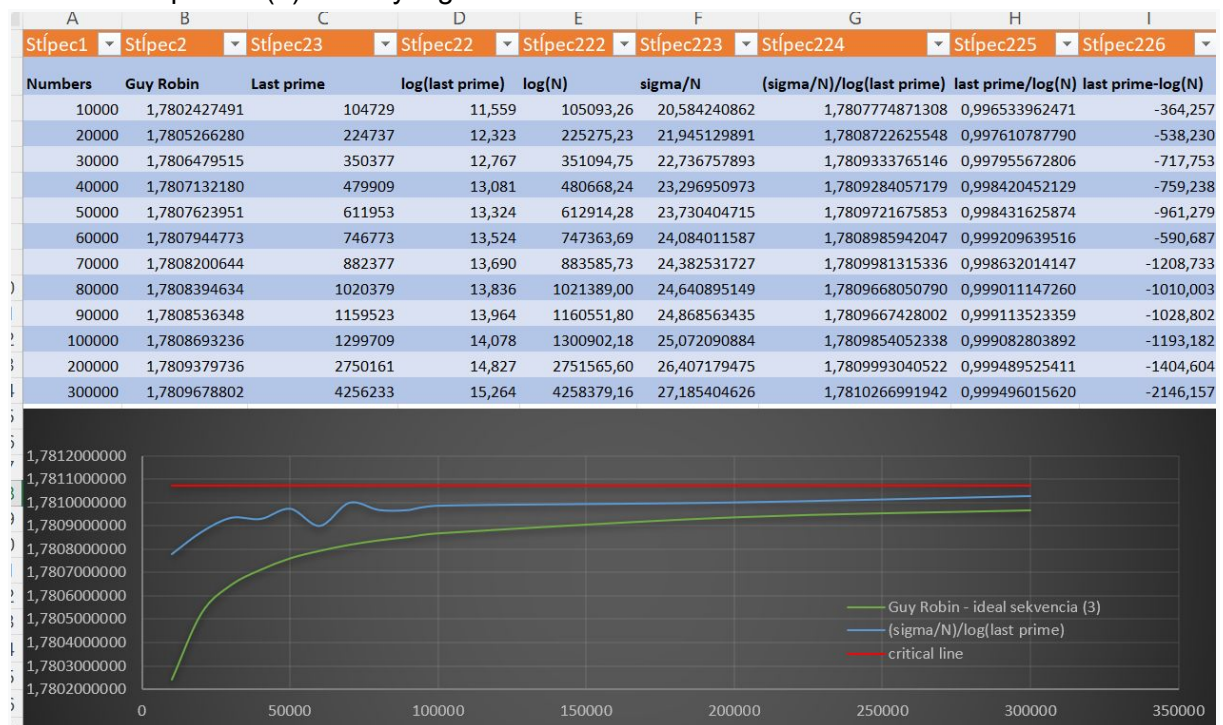
$$\beta(n) := \prod_{p_i \in \text{prime}}^{p_n} \frac{p_i}{p_i-1}; \quad \beta(n) > \sup \frac{\sigma(n)}{n} \quad (2.1)$$

## Reformulation of RH conditions (Robopol)

From empirical testing using the Python programming language, a graph for sequence (1) and sequence (3) is shown. See below. Article link - reference (8).



tab. no.1 sequence (1) for very high numbers.



tab. no.2 sequence(3) for very high numbers.

for sequence (1) tab.no.1:

$$\text{initial test: } n = \prod_{p_1 \in \text{prime}}^{p_{10000}} p_i; \text{ last\_prime} = p_{1000} = 104729$$

$$\text{end test: } n = \prod_{p_1 \in \text{prime}}^{p_{500\,000}} p_i; \text{ last\_prime} = p_{500\,000} = 7368787$$

for sequence (3) - highly composite numbers, tab.no.2:

in terms of the equation (1.7) for highly composite numbers

$$\text{initial test: } n = \sup \prod_{p_1 \in \text{prime}}^{p_{10000}} p_i^{j_i}; \text{ last\_prime} = p_{1000} = 104729$$

$$\text{end test: } n = \sup \prod_{p_1 \in \text{prime}}^{p_{300\,000}} p_i^{j_i}; \text{ last\_prime} = p_{300\,000} = 4256233$$

**Numerical testing shows that the following statement holds:**

$$\text{last\_prime} = p_n; \log(n) \sim p_n \quad (2.2)$$

for sequence (1):

$$\log(n) < \text{last\_prime}; \text{ or } \log(n) < p_n \quad (2.3)$$

for sequence (3) - highly composite numbers:

$$\log(n) > \text{last\_prime}; \text{ or } \log(n) > p_n \quad (2.4)$$

Lagarias theorem:

$$\begin{aligned} \sigma(n) &< \log(n) + \gamma + \varepsilon + e^{\ln(n)+\gamma+\varepsilon} \log(\log(n) + \gamma + \varepsilon) = \\ &= \log(n) + \gamma + \varepsilon + n e^{\gamma} e^{\varepsilon} \log(\log(N) + \gamma + \varepsilon) \end{aligned} \quad (2.5)$$

$$\frac{\sigma(n)}{n} < \frac{\log(n)+\gamma+\varepsilon+n \cdot e^{\gamma} \cdot e^{\varepsilon} \cdot \log(\log(n)+\gamma+\varepsilon)}{n} \quad (2.6)$$

creates a limit for Gronwall theorem, while for  $N \rightarrow \infty, \varepsilon = 0$

$$\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n \log(\log(n))} = \lim_{n \rightarrow \infty} \frac{\log(n)+\gamma+n e^{\gamma} \log(\log(n)+\gamma)}{n \log(\log(n))} = e^{\gamma} \quad (2.7)$$

Guy Robin theorem :

$$\frac{\sigma(n)}{n} < e^{\gamma} \log(\log n); n > 5040 \quad (2.8)$$

Guy Robin theorem is a stronger statement because:

$$e^{\gamma} \log(\log n) < \frac{\log(n)+\gamma+\varepsilon+n e^{\gamma} e^{\varepsilon} \ln(\ln(n)+\gamma+\varepsilon)}{n} \log(\log(n)) \quad (2.9)$$

A stronger theorem arises in this article, and if it can be proved, the remaining Robin, Lagarias theorem, also applies. This theorem is related to highly-composite numbers, where each number "n" is just a highly-composite number.

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Robopol theorem:

for highly composite numbers in terms of equation (1.7), (2.1) we get:

$$\beta(n) < e^\gamma \log(\log n) \quad (3.0)$$

$$\prod_{p_i \in \text{prime}}^{p_n} \frac{p_i}{p_i - 1} < e^\gamma \log(\log(n)); \quad (3.1)$$

for highly composite numbers - n; if  $p_n \geq p_{10}$

for highly composite numbers in terms of equation (2.4), tab. no.2 we get:

$$\prod_{p_i \in \text{prime}}^{p_n} \frac{p_i}{p_i - 1} < e^\gamma \log(p_n) \quad (3.2)$$

for highly composite numbers - n; if  $p_n \geq p_{100}$

**Equation (3.1) and (3.2) is a stronger statement than equation (2.8) because:**

$\beta(n) > \sup \frac{\sigma(n)}{n}$  and  $\log(n) > p_n$  – for highly composite number "n"

## Test Robopol theorem

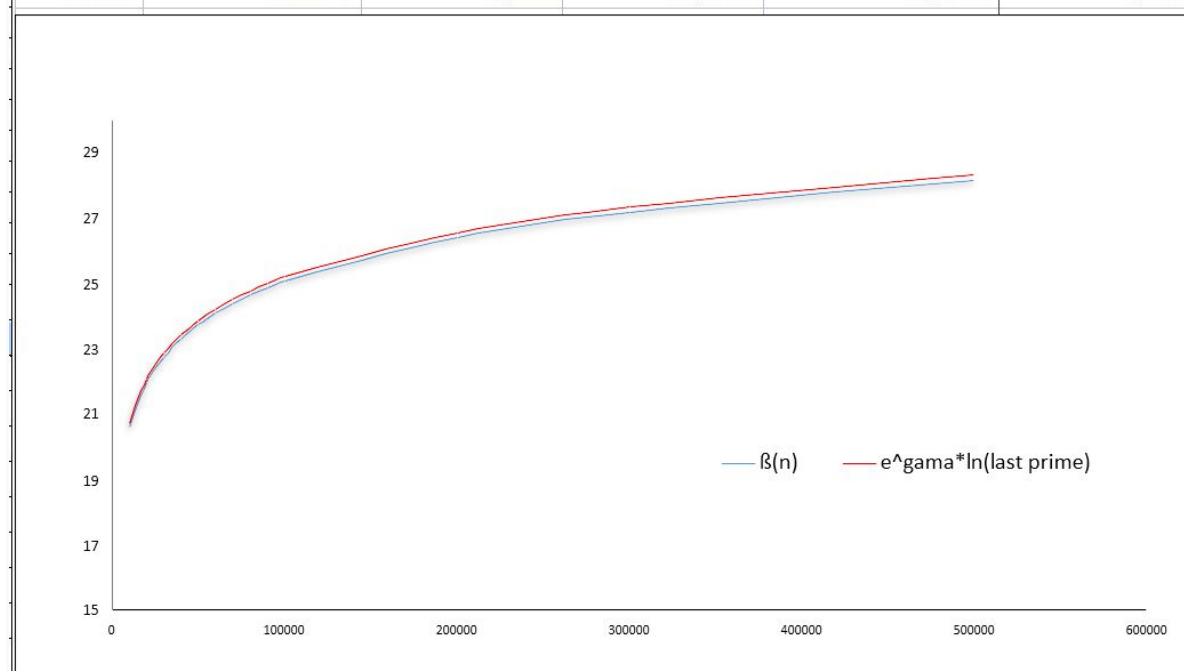
Program (in Python) test Robopol theorem:

reference (10)

Download file in GitHub: [sigma-max\\_test.py](#)

The table no.3 shows the test results for very large numbers:

Numbers	Last prime	$\ln(\text{last prime})$	$\beta(n)$	$e^{\gamma} \ln(\text{last prime})$	
10000	104729	11,55913	20,59352	20,7021	0,10858
20000	224737	12,32269	21,9515	22,0696	0,1181
30000	350377	12,76676	22,74206	22,86494	0,12288
40000	479909	13,08135	23,30135	23,42835	0,127
50000	611953	13,32441	23,73436	23,86366	0,1293
60000	746773	13,52352	24,08757	24,22026	0,13269
70000	882377	13,69037	24,38586	24,5191	0,13323
80000	1020379	13,83568	24,644	24,77934	0,13534
90000	1159523	13,96352	24,87145	25,00829	0,13684
100000	1299709	14,07765	25,07481	25,2127	0,13789
200000	2750161	14,82717	26,40906	26,55507	0,146
300000	4256233	15,2639	27,18694	27,33723	0,15029
500000	7368787	15,81276	28,16442	28,32024	0,15582
1000000	15485863	16,55544	29,48665	29,65035	0,1637
2000000	32452843	17,2953	30,80453	30,97542	0,17089



tab. No. 3 Test robopol theorem for very large numbers.

It can be seen from the table and the graph that the theorem is very tightly fulfilled and they differ slightly towards infinity. This means that the theorem could apply (as a stronger statement compared to Robin's theorem) to infinity. But we need proof of that. Of course, in the sense of equation (3.2) for highly composite numbers that are already large.

## Approximation $\pi(x)$

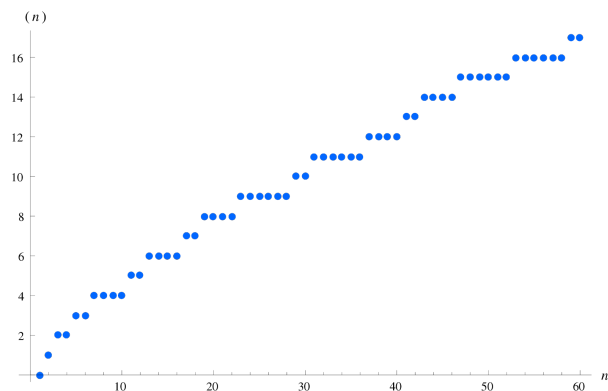


Figure no.2 The values of  $\pi(n)$  for the first 60 positive integers, source: wiki

Prime- counting function:

$$\pi(x) \sim \frac{x}{\log(x)}; \text{ year: 1792} \quad (3.3)$$

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1 \quad (3.4)$$

better approximation  $\pi(x)$ :

$$\pi(x) \sim Li(x) := \int_2^x \frac{dt}{\log(t)} \quad (3.5)$$

$$\lim_{x \rightarrow \infty} Li(x)/\pi(x) = 1 \quad (3.6)$$

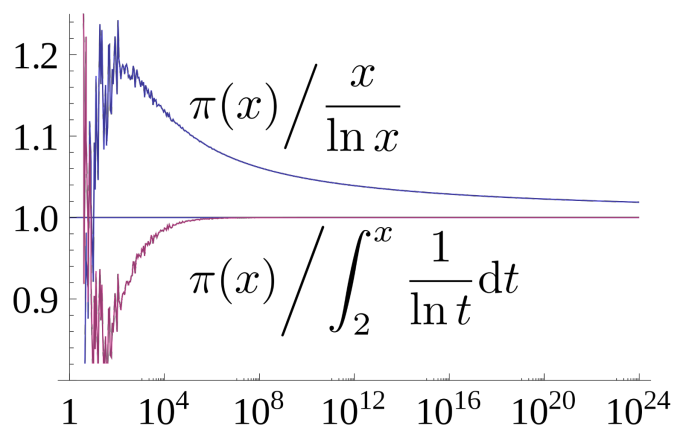


Figure no.3 Graph showing ratio of the prime-counting function  $\pi(x)$  to two of its approximations,  $x/\log x$  and  $Li(x)$ . Source: wiki.

According to source (3):

to read:

In Pierre Dusart's thesis there are stronger versions of this type of inequality that are valid for larger  $x$ . Later in 2010, Dusart proved:

$$\frac{x}{\log(x)-1} < \pi(x) < \frac{x}{\log(x)-1.1} \quad (3.7)$$

The proof by de la Vallée Poussin implies the following: For every  $\varepsilon > 0$ , there is an  $S$  such that for all  $x > S$ ,

$$\frac{x}{\log(x) - (1 - \varepsilon)} < \pi(x) < \frac{x}{\log(x) - (1 + \varepsilon)} \quad (3.8)$$

**Equation (3.8) is very important for subsequent evidence.** That means that the approximation  $\frac{x}{\log(x) - 1}$  is the one that should be roughly equal to  $\pi(x)$  towards infinity. That is, it is a critical boundary, a line.

$$\pi(x) \sim \frac{x}{\log(x) - 1} \quad (3.9)$$

$$\lim_{x \rightarrow \infty} \pi(x) = \lim_{x \rightarrow \infty} \frac{x}{\log(x) - 1} \quad (3.10)$$

example:

$$x = 10^{25}$$

$$\pi(10^{25}) = 176846309399143769411680$$

$$\frac{10^{25}}{\log(10^{25}) - (1 + 0.019)} - \pi(10^{20}) \sim 2.02482 \cdot 10^{18}$$

$$\frac{10^{25}}{\log(10^{25}) - (1 + 0.019)} > \pi(10^{25})$$


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## Context in Robopol's theorem

According to equation (3.2), we derive other contexts that should apply.

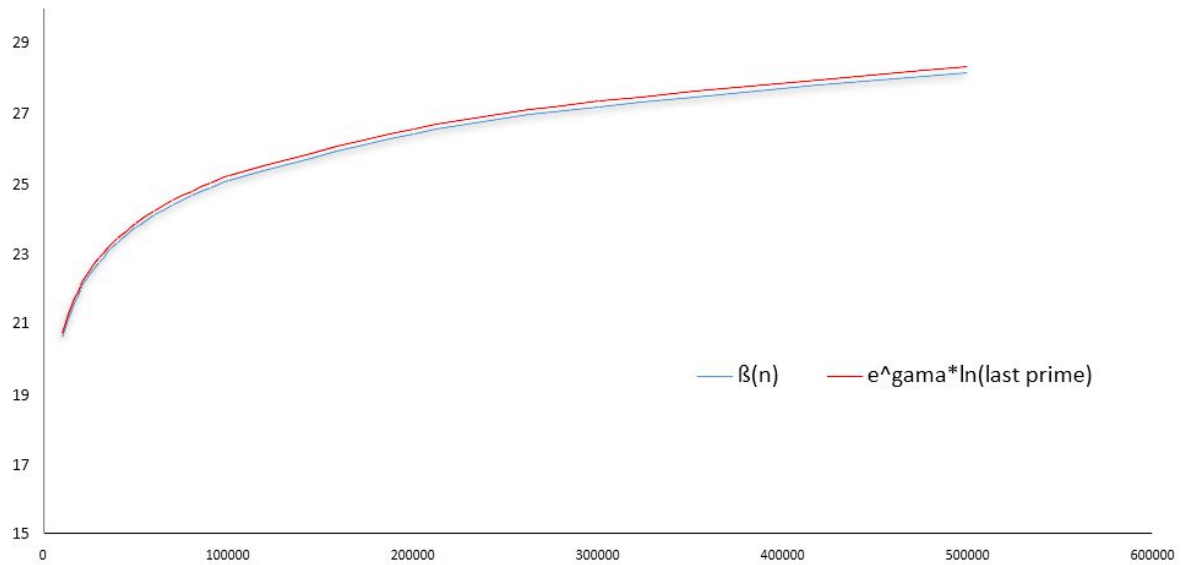


Figure no.4 : Development chart for robopol theorem (for very large numbers).

### Statement (1.0):

Let equation (4.0) hold for the domain of the equation (3.2):

$$\text{for } p_k \geq p_{100}; \quad \beta(p_{k+1}) - \beta(p_k) \sim e^\gamma \log(p_{k+1}) - e^\gamma \log(p_k) \quad (4.0)$$

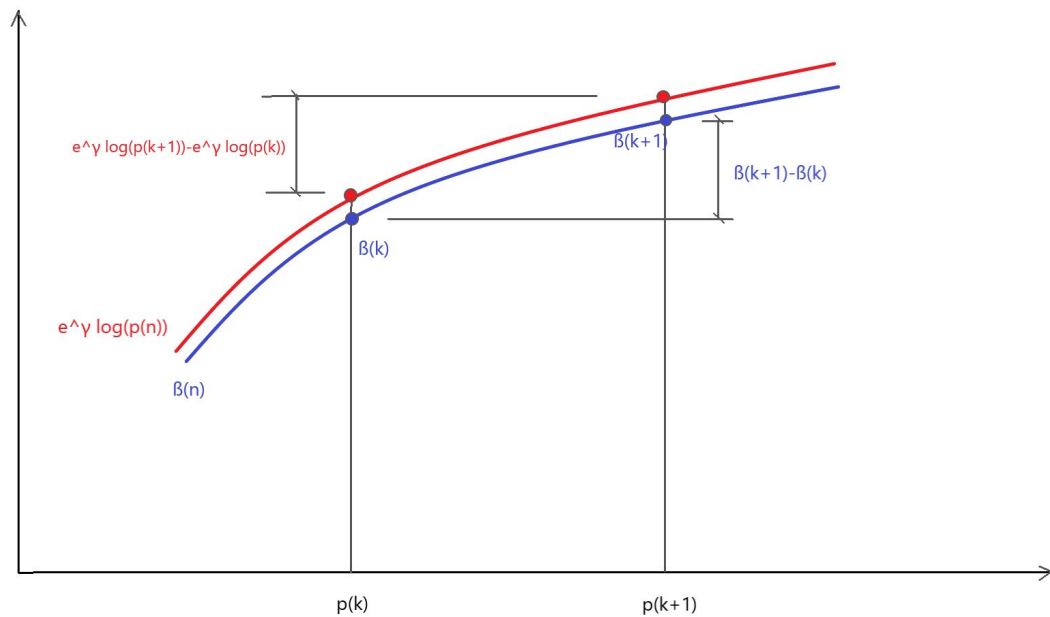


Figure no.5 expression of equation 4.0.

In FIG. No. 5 is the expression of statement (1.0) in geometric interpretation.

modify equation (4.0):

$$\prod_{p_i \in \text{prime}}^{p_{k+1}} \left( \frac{p_i}{p_i-1} \right) - \prod_{p_i \in \text{prime}}^{p_k} \left( \frac{p_i}{p_i-1} \right) \sim e^\gamma \log(p_{k+1}) - e^\gamma \log(p_k) \quad (4.1)$$

$$\varepsilon_k = \prod_{p_i \in \text{prime}}^{p_k} \left( \frac{p_i}{p_i-1} \right); \quad \omega_k = \log(p_k) e^\gamma \quad (4.2)$$

identity first point in  $p_k$ :  $\varepsilon_k = \omega_k$

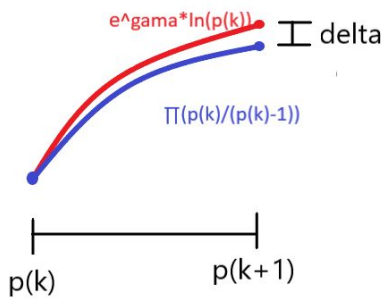


Figure no.6 identification of starting points in  $p_k$ ,  $\varepsilon_k = \omega_k$ .

$$\Delta \varepsilon_k = \varepsilon_k \frac{p_{k+1}}{p_{k+1}-1} - \varepsilon_k = \varepsilon_k \left( \frac{p_{k+1}}{p_{k+1}-1} - 1 \right) \quad (4.3)$$

$$\Delta \omega_k = e^\gamma (\log(p_{k+1}) - \log(p_k)) \quad (4.4)$$

in terms of the equation (4.1):

$$\Delta \omega_k \sim \Delta \varepsilon_k; \quad e^\gamma (\ln(p_{k+1}) - \ln(p_k)) \sim \varepsilon_k \frac{p_{k+1}}{p_{k+1}-1} - \varepsilon_k \quad (4.5)$$

substitute into equation (4.5)  $\varepsilon_k = \omega_k$ , we get:

$$e^\gamma \log(p_{k+1}) - e^\gamma \log(p_k) \sim \log(p_k) e^\gamma \frac{p_{k+1}}{p_{k+1}-1} - \log(p_k) e^\gamma \quad (4.6)$$

$$\log(p_{k+1}) \sim \log(p_k) \frac{p_{k+1}}{p_{k+1}-1} \quad (4.7)$$

example:

$$p_{999\,999\,999} = 22801763477$$

$$p_{1\,000\,000\,000} = 22801763489$$

$$\log(22801763489) \approx \log(22801763477) \cdot \frac{22801763489}{22801763489-1}$$

$$23.850103715924.. \approx 23.850103715442...$$

**Statement (2.0):**

Let there exist a smooth, continuous function  $g(x) = f(x)$ , which approximates  $\pi(x)$  such that the following relation holds towards infinity:

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \pi(x) \quad (4.8)$$

Theorem (2.0) is satisfied in accordance with equations (3.8) and (3.9):

$$g(x) = \frac{x}{\log(x)-1} \quad (4.9)$$

**Statement (3.0):**

Let the following equation hold on the domain of function  $g(x)$  for all  $x > 100$  in the sense of equation (4.7):

$$\log(x + \Delta x) \geq \log(x) \frac{x + \Delta x}{x + \Delta x - 1} \quad (4.10)$$

where:

$\Delta x$  - is the horizontal distance in the sense figure no.7.

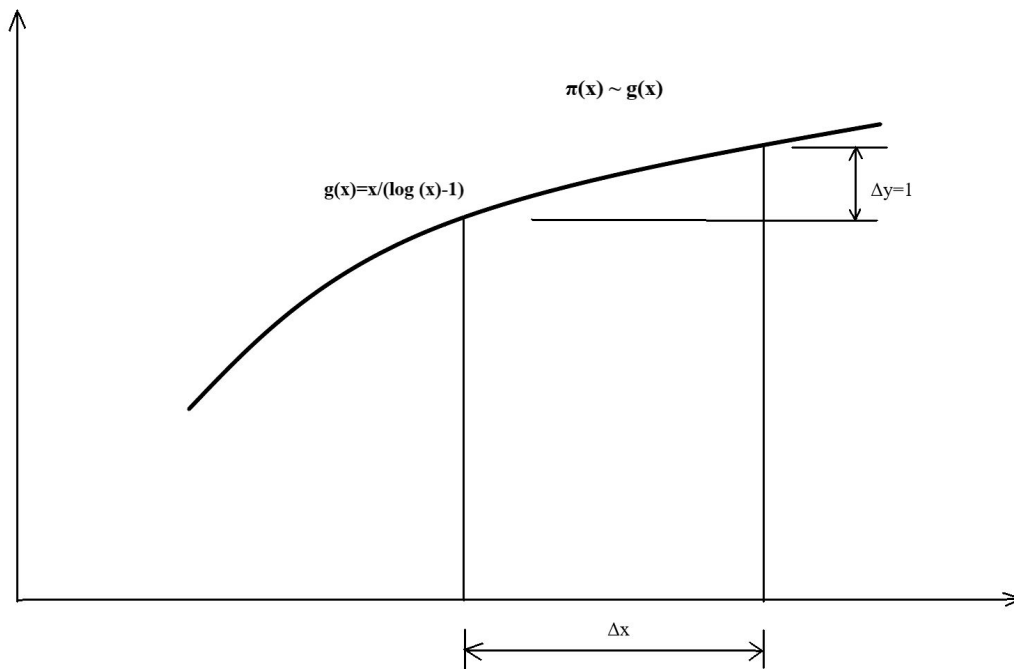


Figure no.7 Horizontal distance  $\Delta x$  in the equation (4.10)

let's define the equation in the sense of the figure no.7:

$$\frac{x}{\log(x)-1} + 1 = \frac{x+\Delta x}{\log(x+\Delta x)-1} \quad (5.0)$$

or

$$\frac{x}{\log(x)-1} = 1 + \frac{x-\Delta x}{\log(x-\Delta x)-1} \quad (5.1)$$

The point of the statements (1.0), (2.0) and (3.0) is that if equation (4.10) is satisfied on the whole domain  $x > 100$ , for all  $x$ , which we calculate from equation (5.1), then the theorem - equation (3.0), (3.2). We will define this in the statement (4.0).

Statement (4.0):

**Eng: If equation (4.10) applies to all  $x > 100$ , where  $\Delta x$  is calculated from equation (5.1), then equation (3.0), (3.2) necessarily also applies.**

Explanation:

Equation (4.10) is relatively simple and intuitive. This equation is already continuous compared to the original (4.7). The original equation (4.7) contained approximately. But that's just because the primes in  $\pi(x)$  contain slight fluctuations from some mean value of the smoothed function that could approximate it perfectly (like a smooth curve, no toothing). Numerical tests (Table no.3) have just shown that in most cases the following were met:

$$\log(p_{k+1}) > \log(p_k) \frac{p_{k+1}}{p_{k+1}-1} \quad (5.2)$$

Thus, to confirm the validity of equation (3.0),(3.2), we examine the validity (5.2) at any point  $p_k$  and its neighbor  $p_{k+1}$ , with the difference that we do this on the smoothed function  $g(x)$  for any  $x > 100$ . But, if we prove the validity for all  $x \rightarrow \infty$ , then it is obvious that equation (3.0) must also hold.

One might argue that  $g(x)$  does not approximate  $\pi(x)$  well. Thus, for small  $x$ ,  $g(x) < \pi(x)$ . But, at the same time we know that the limit  $g(x)$  catches up with  $\pi(x)$ , in the sense of equation (4.8). Thus, if the limit  $g(x)$  is to catch up with  $\pi(x)$ , then it is obvious that towards the infinite - slope  $g(x)$  grows more than  $\pi(x)$ .

From the numerical tests tab. no. 3, however, equation (3.0), (3.2) is tested in large numbers. More precisely, the high-composite numbers for  $p_n \geq p_{100}$  are verified after those listed in tab.no.3. In the following sections, I will therefore try to prove that the key statement (4.0) is indeed valid.

Adjust equation (4.10) as follows:

$$\log(x + \Delta x) \geq \log(x) \frac{x+\Delta x}{x+\Delta x-1}$$

let's make a substitution:

$$x + \Delta x = t$$

$$\log(t) \geq \log(t - \Delta x) \frac{t}{t-1}$$

$$\log(t - \Delta x) \geq \frac{\log(t) \cdot (t-1)}{t}$$

eliminate logarithm:

$$t - \Delta x \geq t^{(t-1)/t}$$

$$\Delta x \geq t - t^{(t-1)/t} \quad (5.3)$$

define  $\Delta x_{min}$ :

$$\Delta x_{min} := t - t^{(t-1)/t} \quad (5.4)$$

$$\Delta x \geq \Delta x_{min} \quad (5.5)$$

derivation of a function:

$$\frac{d}{dt} (t - t^{(t-1)/t}) = t^{-(t+1)/t} (t + \log(t) - 1) \quad (5.6)$$

limit function

$$\lim_{t \rightarrow \infty} t^{-(t+1)/t} (t + \log(t) - 1) = 1 \quad (5.7)$$

example:

$$t = 1000, \Delta x_{min} = 1000 - 1000^{(1000-1)/1000} = 6.883951579..$$

$$x = t - \Delta x_{min} = 1000 - 6.883951579 = 993.116048420..$$

$$\log(x + \Delta x_{min}) = \log(x) \cdot \frac{x + \Delta x_{min}}{x + \Delta x_{min} - 1}$$

$$\log(1000) = \log(993.1160484) \cdot \frac{993.1160484 + 6.883951579}{993.1160484 + 6.883951579 - 1}$$

$$6.907755279... = 6.907755279...$$


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## Approximation analysis $\pi(x) = \frac{x}{\log(x)}$

We first examine the simpler case of approximation  $\pi(x)$ .

In the sense of picture no.7 (for approximation  $\pi(x) = \frac{x}{\log(x)}$ ) we get the equation:

$$\frac{x}{\log x} = 1 + \frac{x - \Delta x}{\log(x - \Delta x)} \quad (6.0)$$

The real analytical solution to the equation is:

$$\Delta x = x - e^{-W_{-1}(\log(x)/(\log(x)-x))}; \quad \text{for } \Delta x > 0 \quad (6.1)$$

or

$$\Delta x = x - \frac{\log(x) - x}{\log x} W_{-1}\left(\frac{\log(x)}{\log(x) - 1}\right); \quad \text{for } \Delta x > 0 \quad (6.2)$$

$W_{-1}(z)$  – is Lambert function, reference (4)

derivácia funkcie / derivation of a function:

$$\frac{d}{dx} \left( \frac{\log(x)-x}{\log x} W_{-1} \left( \frac{\log(x)}{\log(x)-1} \right) \right) = - \frac{(\log(x)-1) W_{-1} \left( \frac{\log(x)}{\log(x)-1} \right)^2}{\log^2(x) (W_{-1} \left( \frac{\log(x)}{\log(x)-1} \right) + 1)} \quad (6.3)$$

limita funkcie / limit function:

$$\lim_{x \rightarrow \infty} - \frac{(\log(x)-1) W_{-1} \left( \frac{\log(x)}{\log(x)-1} \right)^2}{\log^2(x) (W_{-1} \left( \frac{\log(x)}{\log(x)-1} \right) + 1)}$$

answer:

No result found in terms of standard mathematical functions.

According to equation (5.5), for  $x > 100$ , it must hold:

$$\Delta x \geq \Delta x_{min}; \quad x - \frac{\log(x)-x}{\log(x)} \cdot W_{-1} \left( \frac{\log(x)}{\log(x)-x} \right) \geq x - x^{(x-1)/x} \quad (6.4)$$

example(1):

$$x = 10^6$$

$$\Delta x = \frac{\log(10^6)-10^6}{\log(10^6)} \cdot W_{-1} \left( \frac{\log(10^6)}{\log(10^6)-10^6} \right) = 14.89353360214...$$

$$\Delta x_{min} = 10^6 - (10^6)^{(10^6-1)/10^6} = 13.815415124237...$$

$$\Delta x > \Delta x_{min}; \quad \Delta x - \Delta x_{min} = 1.078118477907...$$

example(2):

$$x = 10^{200}$$

$$\Delta x = \frac{\log(10^{200})-10^{200}}{\log(10^{200})} \cdot W_{-1} \left( \frac{\log(10^{200})}{\log(10^{200})-10^{200}} \right) = 461.519194796772488...$$

$$\Delta x_{min} = 10^{200} - (10^{200})^{(10^{200}-1)/10^{200}} = 460.517018598809...$$

$$\Delta x > \Delta x_{min}; \quad \Delta x - \Delta x_{min} = 1.00217...$$

From numerical calculations, we would expect the limits to:

$$\lim_{x \rightarrow \infty} \Delta x - \Delta x_{min} = 1$$

Adjust equation (6.0) to this desired shape:

$$\frac{x}{\log x} - 1 - \frac{x-\Delta x}{\log(x-\Delta x)} = 0 \quad (6.5)$$

next:

$$\frac{x \log(x-\Delta x) - \log(x) \log(x-\Delta x) - \log(x)(x-\Delta x)}{\log(x) \log(x-\Delta x)} = 0$$

next:

$$(x - \log(x)) \log(x - \Delta x) - \log(x)(x - \Delta x) = 0 \quad (6.6)$$

Let's expansion series at  $x = \infty$  :

$$\begin{aligned} & (\Delta x + \Delta x \log(x) - \log^2(x)) - \frac{\Delta x (\Delta x - 2 \log(x))}{2x} + \frac{\Delta x^2 (3 \log(x) - 2 \Delta x)}{6x^2} + \frac{\Delta x^3 (4 \log(x) - 3 \Delta x)}{12x^3} + \\ & \frac{\Delta x^4 (5 \log(x) - 4 \Delta x)}{20x^4} + \frac{\Delta x^5 (6 \log(x) - 5 \Delta x)}{30x^5} + \frac{\Delta x^6 (7 \log(x) - 6 \Delta x)}{42} + O\left(\left(\frac{1}{x}\right)^7\right) \end{aligned} \quad (6.7)$$

for  $x \rightarrow \infty$  the equation is reduced:

$$\Delta x + \Delta x \log(x) - \log^2(x) = 0 \quad (6.8)$$

we get from the equation  $\Delta x$ :

$$\Delta x = \frac{\log^2(x)}{\log(x)-1} \quad (6.9)$$

According to (6.4) we calculate the limit:

$$\lim_{x \rightarrow \infty} \Delta x - \Delta x_{\min} = \lim_{x \rightarrow \infty} \frac{\log^2(x)}{\log(x)-1} - \left(x - x^{(x-1)/x}\right) = 1 \quad (6.10)$$

Result:

Equation (6.10) showed that equation  $\Delta x \geq \Delta x_{\min}$  is true to infinity.

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## Approximation analysis $\pi(x) = \frac{x}{\log(x)-1}$

We now check the approximation  $\pi(x) = \frac{x}{\log(x)-1}$  in the sense of equation (3.9), (3.10).

The solution of equation (5.1) is:

$$\Delta x = \frac{(x - \log(x) + 1) \cdot W_{-1}\left(-\frac{e^{\frac{1+x-\log(x)}{x}} \sqrt{\frac{x+1}{x}} \cdot (\log(x)-1)}{x - \log(x) + 1}\right)}{\log(x)-1} + x \quad (7.0)$$

$W_{-1}(z)$  – is Lambert function, reference (4)

According to equation (5.5), for  $x > 100$ , it must hold:

$$\Delta x \geq \Delta x_{\min};$$

$$\frac{(x-\log(x)+1) \cdot W_{-1}\left(-\frac{\sqrt{\frac{e^{x+1}}{x} \cdot (\log(x)-1)}}{x-\log(x)+1}\right)}{\log(x)-1} + x \geq x - x^{(x-1)/x} \quad (7.1)$$

example:

$$x = 10^{180}$$

$$\Delta x = \frac{(10^{180} - \log(10^{180}) + 1) \cdot W_{-1}\left(-\frac{\sqrt{\frac{e^{10^{180}+1}}{10^{180}} \cdot (\log(10^{180})-1)}}{10^{180} - \log(10^{180}) + 1}\right)}{\log(10^{180})-1} + 10^{180} =$$

$$= 414.467741185201..$$

$$\Delta x_{\min} = 10^{180} - (10^{180})^{(10^{180}-1)/10^{180}} = 414.465316...$$

$$\Delta x > \Delta x_{\min}; \Delta x - \Delta x_{\min} \sim 0$$

From numerical calculations, we would expect the limits to:

$$\lim_{x \rightarrow \infty} \Delta x - \Delta x_{\min} = 0$$

Adjust equation (5.1) to this desired shape:

$$\frac{x}{\log x - 1} - 1 - \frac{x - \Delta x}{\log(x - \Delta x) - 1} = 0 \quad (7.2)$$

next:

$$\frac{x}{\log(x)-1} - \frac{x}{\log(x-\Delta x)-1} + \frac{\Delta x}{\log(x-\Delta x)-1} - 1 = 0 \quad (7.3)$$

Let's expansion series at  $x = \infty$  :

Puiseux series

$$- \frac{2 \Delta x - (\Delta x - 2) \log(x) + \log^2(x) + 1}{(\log(x) - 1)^2} + \frac{\Delta x^2 (\log(x) - 3)}{2 x (\log(x) - 1)^3} + O\left(\left(\frac{1}{x}\right)^2\right) \quad (7.4)$$

for  $x \rightarrow \infty$  the equation is reduced:

$$- \frac{2 \Delta x - (\Delta x - 2) \log(x) + \log^2(x) + 1}{(\log(x) - 1)^2} = 0 \quad (7.5)$$

we get from the equation  $\Delta x$ :

$$\Delta x = \frac{(\log(x) - 1)^2}{\log(x) - 2} \quad (7.6)$$

we calculate the limit:

$$\lim_{x \rightarrow \infty} \Delta x - \Delta x_{min} = \lim_{x \rightarrow \infty} \frac{(\log(x)-1)^2}{\log(x)-2} - (x - x^{(x-1)/x}) = 0 \quad (7.9)$$

Series expansion at  $x=\infty$

$$\frac{1}{\log(x)-2} + \frac{\log^2(x)}{2x} + o\left(\left(\frac{1}{x}\right)^2\right) \quad (7.10)$$

**Result:**

**Equation (7.9) showed that equation  $\Delta x \geq \Delta x_{min}$  is true to infinity.**

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$$\text{Approximation analysis } \pi(x) = \frac{x}{\log(x)-\epsilon}$$

for:

$$\pi(x) = \frac{x}{\log(x)-\epsilon}; \quad \epsilon = 1 + \epsilon; \quad \epsilon > 0 \quad (8.0)$$

The solution of equation (8.0) is:

$$\Delta x = \frac{(x+\log(x)+\epsilon) \cdot W_{-1}\left(\frac{(e^{\epsilon^2/x+\epsilon} \cdot x^{-\epsilon/x})^{x/(x+\epsilon-\log(x))} \cdot (\epsilon-\log(x))}{x-\log(x)+\epsilon}\right)}{\log(x)-\epsilon} + x \quad (8.1)$$

$W_{-1}(z)$  – is Lambert function, reference (4)

According to equation (5.5), for  $x > 100$ , it must hold:

$$\Delta x \geq \Delta x_{min};$$

$$\frac{(x+\log(x)+\epsilon) \cdot W_{-1}\left(\frac{(e^{\epsilon^2/x+\epsilon} \cdot x^{-\epsilon/x})^{x/(x+\epsilon-\log(x))} \cdot (\epsilon-\log(x))}{x-\log(x)+\epsilon}\right)}{\log(x)-\epsilon} + x \geq x - x^{(x-1)/x} \quad (8.2)$$

example(1):

$$x = 10^{15}$$

$$\epsilon = 1.0000000000000001$$

$$\Delta x = \frac{277}{8} = 34.625$$

$$\Delta x_{min} = 10^{15} - (10^{15})^{(10^{15}-1)/10^{15}} = 34.538...$$

$$\Delta x > \Delta x_{min}$$

example(2):

$$x = 10^{16}$$

$$\epsilon = 1.0000000000000001$$

$$\Delta x = 36$$

$$\Delta x_{min} = 10^{16} - (10^{16})^{(10^{16}-1)/10^{16}} = 36.841...$$

$$\Delta x < \Delta x_{min}$$

Nie je splnená rovnica 8.2 /Equation 8.2 is not met.

Result analysis:

Equation (8.2) for the approximation (8.0) will not be generally satisfied. However, the approximation (8.0) **is greater than**  $\pi(x)$  in the sense of equation (3.8).

## Reference:

- (1) [RAMANUJAN, ROBIN, HIGHLY COMPOSITE NUMBERS, AND THE RIEMANN HYPOTHESIS](#)
- (2) <http://math.colgate.edu/~integers/I33/I33.pdf>
- (3) [Prime number theorem](#)
- (4) [Numerical Evaluation of the Lambert W Function](#)
- (5) [In 1984 Guy Robin](#)
- (6) [https://en.wikipedia.org/wiki/Prime-counting\\_function](https://en.wikipedia.org/wiki/Prime-counting_function)
- (7) [web: https://robopol.sk/blog/riemannova-hypotezagoldenpart](https://robopol.sk/blog/riemannova-hypotezagoldenpart)
- (8) [web: https://robopol.sk/blog/riemannova-hypoteza-dodatok](https://robopol.sk/blog/riemannova-hypoteza-dodatok)
- (9) [web: https://robopol.sk/blog/riemannova-hypotezahladanie-dokazu](https://robopol.sk/blog/riemannova-hypotezahladanie-dokazu)
- (10) [web: https://github.com/robopol/Riemann-hypothesis](https://github.com/robopol/Riemann-hypothesis)